Curves

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We will use the following fact without a proof

<u>Thm:</u> Let $f: X \to Y$ be a proper morphism of noeth schemes

For $\mathcal{F} \in Coh(X)$, $R^i f_* \mathcal{F} \in Coh(X), \forall i$

We do not need the full strength of the Thm above, but the following consequence

<u>Thm.</u> A be a noeth ring. $X \to Spec(A)$ proper. Then

for $\mathcal{F} \in Coh(X)$, $H^i(X, \mathcal{F})$ is a finitely generated A mod $\forall i$. In particular $H^0(X, \mathcal{F})$ is a f.g A-mod.

<u>Thm:</u> Let k be an algebraically closed field. $X \to Spec(k)$ is proper. $\mathcal{L}$ be an invertible $\mathcal{O}_X$-mod. Given $s_0, s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$, Denote The k subspace of $\Gamma(X, \mathcal{L})$ spanned by $s_0 \ldots, s_n$ by V

① The map $\varphi: X \to \mathbb{P}^n_k$ given by $s_0 \ldots s_n$ is a closed immersion.

② ⓐ For two distinct closed pts $P, Q \in X$, $\exists\, s \in V$ s.t $s_P \in m_P \mathcal{L}_P$ but $s_Q \notin m_Q \mathcal{L}_Q$.

ⓑ Given a closed pt $P \in V$, Let $V_P = \{ s \in V \mid s_P \in m_P \mathcal{L}_P \}$

Then the subspace of $\frac{m_P \mathcal{L}_P}{m_P^2 \mathcal{L}_P}$ spanned by the image of V_P is $\frac{m_P \mathcal{L}_P}{m_P^2 \mathcal{L}_P}$.

<u>Pf.</u> ① $\Rightarrow$ (2, a) Given such P, Q, $\varphi(P) \neq \varphi(Q)$

$\Rightarrow \exists$ a hyperplane $H = a_0 x_0 + \cdots + a_n x_n$, $a_i \in k$ such that H vanishes at $\varphi(P)$ but not at $\varphi(Q)$

Realizing $H \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$, This means $H_P \in m_{\varphi(P)} \mathcal{O}(1)_{\varphi(P)}$

Realizing $H \in \Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$, This means $H_P \in m_{\varphi(P)} \mathcal{O}(1)_{\varphi(P)}$ but $H_Q \notin m_{\varphi(Q)} \mathcal{O}(1)_{\varphi(Q)}$.

Recall That $\varphi : X \to \mathbb{P}^n_k$ induces an isom

$$\varphi^* \mathcal{O}(1) \xrightarrow{\sim} \mathcal{L} \quad \text{sending } \varphi^*(x_i) \text{ to } s_i$$

For any closed pt $x \in X$, The above isom induces an isom

$$\frac{\varphi^* \mathcal{O}(1)_x}{m_x \varphi^* \mathcal{O}(1)_x} \xrightarrow{\sim} \frac{\mathcal{L}_x}{m_x \mathcal{L}_x}$$

$\ast$

$$\uparrow \wr$$

$$\frac{\mathcal{O}(1)_{\varphi(x)} \otimes \mathcal{O}_{X,x}}{m_x(\mathcal{O}(1)_{\varphi(x)} \otimes \mathcal{O}_{X,x})}$$

$$\uparrow \wr \quad \leftsquigarrow \mathcal{O}_{X,x} \otimes (0 \to m_{\varphi(x)} \mathcal{O}(1)_{\varphi(x)} \to \mathcal{O}(1)_{\varphi(x)} \to \frac{\mathcal{O}(1)_{\varphi(x)}}{m_{\varphi(x)} \mathcal{O}(1)_{\varphi(x)}} \to 0$$

$\&$ $m_{\varphi(x)} \mathcal{O}_{X,x} = m_x$ as $\varphi_x : \mathcal{O}_{\mathbb{P}^n, \varphi(x)} \to \mathcal{O}_{X,x}$ is surj.

$$\frac{\mathcal{O}(1)_{\varphi(x)}}{m_{\varphi(x)} \mathcal{O}(1)_{\varphi(x)}} \otimes \mathcal{O}_{X,x}$$

So a section $t \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ vanishes at $\varphi(x)$ $\Leftrightarrow$ $\varphi^*(t)$ vanishes at x.

$\therefore$ $\varphi^*(H)$ does The job.

(1) $\Rightarrow$ (2, b)
$\exists$ $0 \le i \le n$ s.t $(s_i)_P \notin m_P \mathcal{L}_P$.

Recall That $\varphi^{-1}(D_+(x_i)) = D_{s_i}$ and

$\varphi : D_{s_i} \to D_+(x_i)$ is induced by The k-alg homo

$$\psi_i : k\left[\frac{x_0}{x_i}, \ldots \widehat{\frac{x_i}{x_i}}, \ldots \frac{x_n}{x_i}\right] \longrightarrow \Gamma(D_{s_i}, \mathcal{O}_X)$$

$$x_j/x_i \longmapsto s_j/s_i$$

φ closed immersion $\Rightarrow$ ψ_i is surj.

$\Rightarrow$ The map induced by ψ_i

$\Rightarrow$ The map induced by φ:

$$m_{\varphi(P)} \longrightarrow \frac{m_{\varphi(P)}}{m^2_{\varphi(P)}} \longrightarrow \frac{m_P}{m_P^2} \text{ is surjective}$$

Since k is alg closed $m_{\varphi(P)} = \left(\frac{x_0}{x_i} - \lambda_0, \dots, \frac{x_n}{x_i} - \lambda_n\right)$

$\Rightarrow$ The images of $\underset{\parallel}{\frac{s_0}{s_i}} - \lambda_0, \dots, \underset{\parallel}{\frac{s_n}{s_i}} - \lambda_n$ generate m_P/m_P^2

$$\frac{s_0 - \lambda_0 s_i}{s_i}, \dots, \frac{s_n - \lambda_n s_i}{s_i}$$

$\Rightarrow (s_j - \lambda_j s_i)_P \in m_P \mathcal{L}_P \quad \forall j \neq i$
and generate $\frac{m_P \mathcal{L}_P}{m_P^2 \mathcal{L}_P}$

(2) $\Rightarrow$ (1)

Step 1: $\varphi: X \longrightarrow \mathbb{P}^n_k$ is a homeo onto its image:

X is proper, so φ is closed, so its enough to prove That $|\varphi|: |X| \to |\mathbb{P}^n_k|$ is injective.

For That note, for two closed pts $P \neq Q$, $\varphi(P) \neq \varphi(Q)$

Indeed 2.a allows us to choose

$$s = a_0 s_0 + \dots + a_n s_n \quad \text{s.t} \quad s_P \in m_P \mathcal{L}_P \text{ but } s_Q \notin m_Q \mathcal{L}_Q$$

Set $H = \sum a_i x_i$, $\varphi^* H = s$ under The isom $\varphi^* \mathcal{O}(1) \simeq \mathcal{L}$ sending $\varphi^*(x_i)$ to s_i

So $\Rightarrow H_{\varphi(P)} \in m_{\varphi(P)} \mathcal{O}(1)_{\varphi(P)}$, but $H_{\varphi(Q)} \notin m_{\varphi(Q)} \mathcal{O}(1)_{\varphi(Q)}$.

So $\varphi(P) \neq \varphi(Q)$

Now we show That injectivity on The closed pts $\Rightarrow$ injectivity.

Consider The map

$$X \times_k X \xrightarrow{(\varphi, \varphi)} \mathbb{P}^n \times \mathbb{P}^n$$

$$X \times_k X \longrightarrow \mathbb{P}^n \times \mathbb{P}^n$$

If φ is not injective, the fibre product $\Delta_\varphi \neq \Delta_X$

$$\begin{tikzcd} & X \times_k X \arrow[r] & \mathbb{P}^n \times \mathbb{P}^n \\ X \arrow[ur, "\Delta \text{ diagonal}"] \arrow[r] & \Delta\varphi \arrow[u] \arrow[r] & \mathbb{P}^n \arrow[u, "\Delta"'] \end{tikzcd}$$

Indeed, for $P \neq Q$ in X if $\varphi(P) = \varphi(Q)$, (P, Q) gives a pt in $\Delta\varphi \setminus \Delta$. Choose a closed pt P_0 in $\Delta\varphi \setminus \Delta$. Since k is alg closed P_0 is a closed pt in $\Delta\varphi$.

Since X is separated, both $\Delta\varphi$ and $\delta(X)$ are closed subschemes of $X \times_k X$. Thus P_0 is closed in $X \times_k X$, which is in $\Delta\varphi$ but not in Δ. Again since k is alg closed closed pts of $X \times X$ are $X(\bar{k}) \times X(\bar{k})$. Let $P_0 = (P, Q)$ where P, Q are closed in X, $P \neq Q$

Since $P_0 \in \Delta\varphi$, $\varphi(P) = \varphi(Q)$, contradicting the injectivity of φ at closed pts.

Now, we show

(†) $(\mathcal{O}_{\mathbb{P}^n})_y \longrightarrow (\varphi_* \mathcal{O}_X)_y$ is surj at every closed pt $y \in \mathbb{P}^n$.

W.L.O.G $y \in \varphi(X)$. Let $\varphi(x) = y$,

Since φ is a homeo onto the image

$$(\varphi_* \mathcal{O}_X)_{\varphi(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$$

2.b $\Rightarrow$ $\dfrac{m_{\varphi(x)}}{m^2_{\varphi(x)}} \longrightarrow \dfrac{m_x}{m_x^2}$ is surj (††)

Since φ is proper, $\varphi_* \mathcal{O}_X$ is coh $\Rightarrow$ $\mathcal{O}_{X,x}$ is a f.g

$(\mathcal{O}_{\mathbb{P}^n})_{\varphi(x)}$ module. (††) $\Rightarrow$ $m_x = \operatorname{Im}(m_{\varphi(x)}) + m_x^2$

Now Nakayama $\Rightarrow$ $m_x = \operatorname{Im} m_{\varphi(x)} = m_{\varphi(x)} \cdot \mathcal{O}_{X,x}$

Since $k \xrightarrow{\sim} \dfrac{\mathcal{O}_{\mathbb{P}^n, \varphi(x)}}{m_{\varphi(x)} \mathcal{O}_{\mathbb{P}^n, \varphi(x)}} \longrightarrow \dfrac{\mathcal{O}_{X,x}}{m_x \mathcal{O}_{X,x}} = \dfrac{\mathcal{O}_{X,x}}{m_{\varphi(x)} \mathcal{O}_{X,x}} \xrightarrow{\sim} k$

is surj, we get $\mathcal{O}_{X,x} = \operatorname{Im}(\mathcal{O}_{\mathbb{P}^n, \varphi(x)}) + m_{\varphi(x)} \mathcal{O}_{X,x}$

Thm Nakayama $\Rightarrow$ $\mathcal{O}_{X,x} = \mathrm{Im}(\mathcal{O}_{\mathbb{P}^n,\varphi(x)})$ $\square$.

Def: k be a field, X/k be a proper scheme. For $\mathcal{F} \in \mathrm{Coh}$, The Euler characteristic of $\mathcal{F}$ is

$$\chi_{\mathcal{F}} = \chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathcal{F})$$

$$= \dim_k H^0(X,\mathcal{F}) - \dim_k H^1(X,\mathcal{F}) + \cdots + (-1)^{\dim(X)} \dim_k H^{\dim(X)}(X,\mathcal{F})$$

Prop. $X \to \mathrm{Spec}(k)$ be proper. Given a short exact seq of coherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

we have $\chi_{\mathcal{F}} = \chi_{\mathcal{F}'} + \chi_{\mathcal{F}''}$

Riemann - Roch on curves.

Let k be an algebraically closed field. By a curve over k, we mean a one dimensional scheme. When $X \to \mathrm{Spec}\, k$ is a regular proper, integral curve, for $f \in k(X)$, we showed

$$\deg(\mathrm{div}(f)) = 0$$

Thus we have a well defined degree map

$$\deg : \mathrm{Cl}(X) = \mathrm{CaCl}(X) = \mathrm{Pic}(X) \longrightarrow \mathbb{Z}$$

$$\deg\left(\sum n_i D_i\right) = \sum n_i$$

Thm: (Riemann - Roch) Let k be an algebraically closed field. $X \to \mathrm{Spec}\, k$ be a regular, proper, integral curve. For a divisor $D \in \mathrm{Weil}(X)$, we have

$$\chi_{\mathcal{O}_X(D)} = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D))$$

$$= \deg D + \chi_{\mathcal{O}_X}$$

$$= \deg D + \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X).$$

Rmk. If $\mathcal{O}_X(D)$ is ample, we know $H^1(X, \mathcal{O}_X(nD)) = 0$ $\forall n >> 0$.

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Then for $n >> 0$, $\dim_k H^0(X, \mathcal{O}_X(D))$

$$= n \cdot \deg D + \chi_{\mathcal{O}_X}$$

Pf of RR: When $D = 0$, the statement is obvious.

We show that for any $D \in \mathrm{Weil}(X)$ and closed $p \in X$, the statement holds for $D \Leftrightarrow$ it holds for $D - P$.

Since any divisor on X can be obtained from 0 by adding and subtracting closed pts, we will be done.

Consider the exact seq

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to i_* k(P) \to 0 \qquad (+)$$

[Note $\mathcal{O}_X(-P)$ is the ideal sheaf defining the closed subscheme $(P, k(P))$. Here $i: (P, k(P)) \to X$ is the closed immersion].

$(+) \otimes \mathcal{O}_X(D)$ gives

$$0 \to \mathcal{O}_X(D-P) \to \mathcal{O}_X(D) \to i_*(k(P) \otimes i^* \mathcal{O}_X(D)) \to 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \wr$$
$$\qquad\qquad\qquad\qquad\qquad\qquad i_*(k(P))$$

Then $\chi_{\mathcal{O}_X(D)} = \chi_{\mathcal{O}_X(D-P)} + \chi_{i_* k(P)}$

Note that $\chi_{i_* k(P)} = 1$ since there is no H^1.

Since $\deg D = \deg(D - P) + 1$, we are done.

Serre duality: (SD)

We will assume the following theorem for curves

Thm (SD) k be an algebraically closed field. $Y \to \mathrm{Spec}(k)$ be a regular, integral, proper scheme $/ k$.

Set $\omega_{Y/k} = \omega_Y = \Lambda^{\dim Y} \Omega_{Y/k}$ (since $\Omega_{Y/k}$ is locally free of rk $= \dim Y$, ω_Y is an invertible sheaf)

Set $\omega_{Y/k} = \omega_Y = \Lambda^{\dim Y}\Omega_{Y/k}$ (since $\Omega_{Y/k}$ is locally free of rk = dim Y, ω_Y is an invertible sheaf)

For every i and any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(Y)$ we have an isom

$$\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y) \xrightarrow{\sim} \mathrm{Hom}_k(H^{\dim Y - i}(Y, \mathcal{F}), k)$$

which is functorial in $\mathcal{F}$.

<u>Rmk.</u> Recall $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{F}, \omega_Y)$ is the i-th right derived functor of

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, -) : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Ab\,gp}$$

We will need the following special case of the Serre duality

<u>Thm.</u> k - alg closed, $X \to \mathrm{Spec}(k)$ is a proper, integral, regular curve. Then for $D \in \mathrm{Weil}(X)$

- $$H^1(X, \mathcal{O}_X(D)) \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(D), \omega_X), k)$$

So
- $$H^1(X, \mathcal{O}_X(D)) \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X \otimes \mathcal{O}_X(-D)), k)$$
$$\simeq \mathrm{Hom}_k(H^0(X, \omega_X \otimes \mathcal{O}_X(-D)), k)$$

- $$\dim_k H^1(X, \mathcal{O}_X(D)) = \dim_k H^0(X, \omega_X \otimes \mathcal{O}_X(-D))$$

<u>Rmk.</u> SD $\Rightarrow$ for a curve X as in the Thm above
$\dim_k H^1(X, \mathcal{O}_X) = \dim_k H^0(X, \omega_X)$

This quantity is called the genus of the curve.

From now on assume $X \to \mathrm{Spec}(k)$ is an integral, regular, proper curve over $\mathrm{Spec}(k)$ and k is alg closed.

<u>Prop:</u> Let $D \in \mathrm{Weil}(X)$ be such that $H^0(X, \mathcal{O}_X(D)) \neq 0$. Then $\deg D \geq 0$.

<u>Pf.</u> Take $0 \neq f \in H^0(X, \mathcal{O}_X(D))$. Then $f \in K(X)$ such that

$$\mathrm{div}(f) + D \text{ is effective.}$$

But $\deg(D) = \deg(\mathrm{div}(f) + D) \geq 0$.

But $\deg(D) = \deg(\operatorname{div}(f) + D) \geq 0$.

Prop. If $\deg D > 0$, $\exists\, n_0$ s.t. $\forall\, n \geq n_0$

$$H^1(X, \mathcal{O}_X(nD)) = 0$$

Pf ω_X is invertible, since X is reg $\exists\, K_X \in \mathrm{Weil}(X)$ such that $\mathcal{O}_X(K_X) \xrightarrow{\sim} \omega_X$.

Take $n_0 = \max\{0, \deg \omega_X\} + 1$

For $n \geq n_0$ since $\deg(K_X - nD) < 0$,

$$H^0(X, \mathcal{O}_X(K_X - D)) = H^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = 0$$

Rest follows by applying SD.

Thm: 1) X be as above. Let $D \in \mathrm{Weil}(X)$ s.t. that $\deg(D) > \max\{\deg \omega_X, 0\} + 3$. Then the map given by an ordered basis of $H^0(X, \mathcal{O}_X(D))$ from X to some projective space is a closed immersion.

In particular D is very ample

(2) Every regular, integral curve which is proper over an algebraically closed field is projective.

Pf. (1) $\Rightarrow$ (2): Indeed consider the divisor $D' = P$, where P is some closed pt on X.

Then $D = (\max\{\deg \omega_X, 0\} + 3) \cdot P$ is very ample on X.

(2) Note,

Claim: For a closed pt $x \in X$, and an invertible sheaf $\mathcal{L}$, and $n \in \mathbb{N}$

$$V_x^n = \{ s \in \Gamma(X, \mathcal{L}) \mid s_x \in m_x^n \mathcal{L}_x \} = H^0(X, \mathcal{L} \otimes \mathcal{O}_X(-nx))$$

Pf. The cokernel of the inclusion

$$0 \to \mathcal{O}_X(-nx) \to \mathcal{O}_X$$ is supported at $\{x\}$ and is isom to the pushforward of $\frac{\mathcal{O}_{X,x}}{m_x^n}$ via the inclusion $i : x \longrightarrow X$.

The exact seq

$$0 \to \mathcal{O}_X(-nx) \to \mathcal{O}_X \to i_* \frac{\mathcal{O}_{X,x}}{m_x^n} \to 0$$

induces an exact seq

$$0 \to \mathcal{L} \otimes \mathcal{O}_X(-nx)) \to \mathcal{L} \to i_*\left(\frac{\mathcal{O}_{X,x}}{m_x^n} \otimes i^*\mathcal{L}\right) \to 0$$

$$\uparrow\wr$$

$$i_*\left(\frac{\mathcal{L}_x}{m_x^n \mathcal{L}_x}\right)$$

Taking cohomology, we get

$$0 \to H^0(X, \mathcal{L} \otimes \mathcal{O}_X(-nx)) \to H^0(X,\mathcal{L}) \to H^0\left(x, \frac{\mathcal{L}_x}{m_x^n \mathcal{L}_x}\right)$$

$$\| \qquad \frac{\mathcal{L}_x}{m_x^n \mathcal{L}}$$

Since $V_x^n = \ker\left(H^0(X,\mathcal{L}) \longrightarrow \frac{\mathcal{L}_x}{m_x^n \mathcal{L}}\right)$ we are done.

Now given a $D \in \mathrm{Weil}(X)$ as in (1) and two closed pts P, Q not necessarily distinct

$$\deg(D - P - Q) > \max\{\deg \omega_X, 0\} + 1$$

If $K_X \in \mathrm{Weil}(X)$ such that $\mathcal{O}_X(K_X) \simeq \omega_X$,

$\deg(K_X - (D - P - Q)) < 0$. By SD,

$$H^1(X, \mathcal{O}_X(D - P - Q)) \overset{SD}{\cong} H^0(X, \omega_X(P + Q - D)) = 0$$

For the same reason $H^1(X, \mathcal{O}_X(D - P)) = H^1(X, \mathcal{O}_X(D))$
$= 0$.

By Riemann-Roch, we get for $D' \in \{D-P-Q, D-P, D-Q, D\}$

$$\dim_k H^0(X, \mathcal{O}_X(D')) = \deg D' + \chi_{\mathcal{O}_X}$$

In particular

$$\dim_k H^0(X, \mathcal{O}_X(D)) \overset{(+)}{=} \dim_k H^0(X, \mathcal{O}_X(D-P)) + 1$$
$$\overset{(++)}{=} \dim_k H^0(X, \mathcal{O}_X(D-P-Q)) + 2$$

- When $P \neq Q$, $(++) \Rightarrow \exists\ s \in H^0(X, \mathcal{O}_X(D-P))$ but not in $H^0(X, \mathcal{O}_X(D-P-Q))$. Realizing $s \in H^0(X, \mathcal{O}_X(D))$ we get a section of $\mathcal{O}_X(D)$ s.t $s_P \in m_P \mathcal{O}_X(D)_P$ but $s_Q \notin m_Q \mathcal{O}_X(D)_Q$.

- When $P = Q$

$(++) \Rightarrow \exists\ s \in H^0(X, \mathcal{O}_X(D))$ s.t $s_P \in m_P \mathcal{O}_X(D)_P$ but $s_P \notin m_P^2 \mathcal{O}_X(D)_P$

Since $\dim_k \dfrac{m_P \mathcal{O}_X(D)_P}{m_P^2 \mathcal{O}_X(D)_P} = \dim_k m_P/m_P^2 = 1$, we conclude

The global section of $\mathcal{O}_X(D)$ vanishing at P spans

$$\frac{m_P \mathcal{O}_X(D)_P}{m_P^2 \mathcal{O}_X(D)_P}$$

So the map given by an ordered basis of $H^0(X, \mathcal{O}_X(D))$ must be a closed embedding.